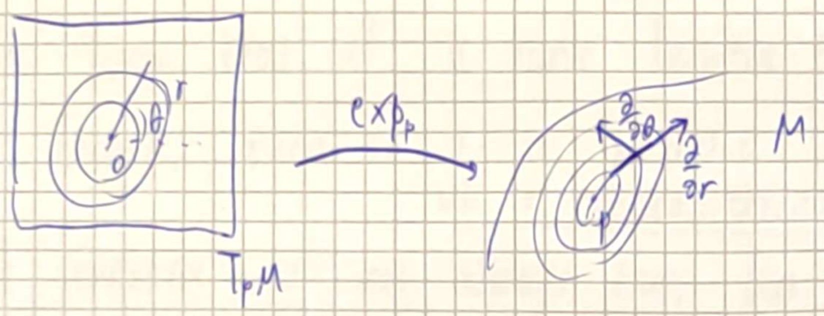
Riemannian Geometry $\boxed{\text{Week 8}}$

Recall from last time:

- We defined the exponential map $\exp_p : \Omega_p \subseteq T_pM \to M$
- Injectivity radius: $i(p)$: $\exp_p$ diffeo on $B_r(0)$ $\forall\ r < i(p)$

Polar coordinates around p: When $\dim M = 2$; If $\{e_1, e_2\}$ orthonormal basis of $(T_pM, g|_p)$.



$q = (r, \theta)$ if $q = \exp_p\left(r(\cos\theta\, e_1 + \sin\theta \cdot e_2\right)$

Valid for $r < i(p)$.

- The curve $\theta = \text{const}$: Geodesic through p

And $\left.\frac{\partial}{\partial r}\right|_{\exp_p(v)} = \frac{\partial}{\partial r}\left(\exp_p\left(r(\cos\theta e_1 + \sin\theta e_2)\right)\right)$

$= d\exp_p|_v(\cos\theta \cdot e_1 + \sin\theta \cdot e_2) = d\exp_p|_v\left(\frac{v}{\|v\|}\right)$

$\left.\frac{\partial}{\partial\theta}\right|_{\exp_p(v)} = \frac{\partial}{\partial\theta}\left(\quad - \quad - \quad \right)$

$= d\exp_p|_v\left(r \cdot (-\sin\theta\, e_1 + \cos\theta \cdot e_2)\right)$

Gauss's lemma: $\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1$, $\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle = 0$

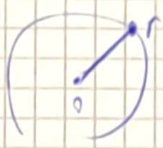
So in polar coordinates:

- $g = dr^2 + (b(r,\theta))^2 \, d\theta^2$, $\quad b(r,\theta) \xrightarrow{r\to 0} 0$, $\frac{\partial b}{\partial r} \xrightarrow{r \to 0} 1$

- The curve ~~[illegible]~~ is a <u>geodesic</u>
  $\gamma(r) = (r, \theta)$

  with $\|\dot\gamma\| = \|\frac{\partial}{\partial r}\| = 1$

  So $\int_0^r \|\dot\gamma\| \, ds = r$



We know that if a curve between two points has minimum length, then it is a geodesic (from 1st variation formula)
<u>up to reparametrization</u>

↖ Need a parametrization with constant speed

The converse is not always true, but:

<u>Proposition</u>: Let $p, q \in (M, g)$ and $d(p,q) < i(p)$.
If $\gamma : [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$ is such that $l(\gamma) = d(p,q)$, then γ is a radial geodesic from p up to reparametrization

<u>Proof</u>: We will only prove the above in dim 2. Let $d = d(p,q)$
Let us fix polar coordinates (r, θ) around p (defined for $r < i(p)$). In these coordinates:

$$g = dr^2 + b^2(r,\theta) \, d\theta^2$$

If $\gamma(t) = (r(t), \theta(t))$:

For any $t_0 \in (0,1]$ such that $\gamma|_{[0,t_0]}$ stays inside the domain of the polar coordinate chart:

$$l(\gamma|_{[0,t_0]}) = \int_0^{t_0} \sqrt{\|\dot\gamma\|^2}\, dt = \int_0^{t_0} \sqrt{(\dot r)^2 + b^2 \cdot (\dot\theta)^2}\, dt \geq$$

$$\geq \int_0^{t_0} \sqrt{(\dot r)^2}\, dt$$

$$\geq r(t_0) - \cancelto{0}{r(0)}$$

So since $l(\gamma|_{[0,1]}) < i(p)$: γ <u>cannot</u> escape the domain of the polar coordinate chart $\Rightarrow q = \gamma(1)$ belongs there

and $d(p,q) = l(\gamma|_{[0,1]}) \geq r(1) =$ length of radial geodesic connecting p to q

$\geq d(p,q)$

So "$=$" holds $\Rightarrow \dot\theta(t) = 0$

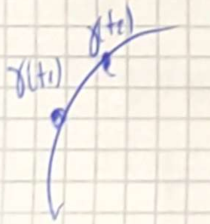
So γ coincides with the $\{\theta = \text{const}\}$ curve $\boxed{}$

<u>Corollary</u>: If $R < i(p)$: $B(p,R) = \exp_p\{v \in T_pM : \|v\| \leq R\}$

and $\partial B(p,R) = \exp_p\{v \in T_pM : \|v\| = R\} = \{q \in M : d(p,q) = R\}$

<u>Def</u>: A curve $\gamma : I \to (M,g)$ locally minimize the length if $\forall t_0 \in I$, $\exists \varepsilon > 0$ such that for all $t_0 - \varepsilon \leq t_1 \leq t_2 \leq t_0 + \varepsilon$

$$d(\gamma(t_1), \gamma(t_2)) = l(\gamma|_{[t_1,t_2]})$$



So: <u>Corollary</u>: A curve locally minimizes the length if and only if it is a reparametrization of a geodesic

What can we say about the geodesic flow on Riemannian manifolds that are complete as metric spaces?

In what follows, (M, g) is connected

Defn: Let (M, g) be a Riem. manifold.

1) We will say that a maximal geodesic $\gamma : I \to M$ is complete if $I = \mathbb{R}$.

2) We will say that (M, g) is geodesically complete if every maximal geodesic is complete.

3) (M, d_g) is a complete metric space if every Cauchy sequence converges

$\boxed{\text{Theorem (Hopf - Rinow)}}$

The following are equivalent for a connected (M, g):

a) (M, g) is geod. complete

b) $\exists\, p \in M$ s.t every maximal geodesic through p is complete

c) Every closed ~~metric~~ ball in the metric space (M, d_g) is compact

"Bolzano-Weierstrass" property

d) (M, d_g) is a complete metric space.

We will prove first the following intermediate result:

Proposition:

Let (M, g) be connected and $p \in M$. If every maximal geodesic through p is complete, then $\forall q \in M$: $\exists$ geodesic from p to q of length $= d(p, q)$

Rmk: The hypothesis is equivalent to saying that $\Omega_p = T_pM$.

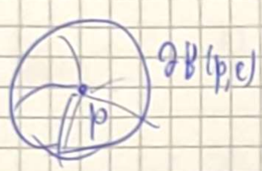
as a corollary: If $\Omega_p = T_pM$, $\exp_p$ is onto.

Proof: Let $d = d(p,q) > 0$ and choose ε such that

$$0 < \varepsilon < \min\{i(p), d(p,q)\}.$$

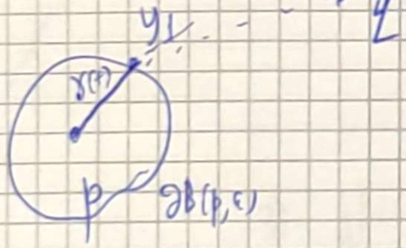
Note that $\exp_p$ is a diffeo on $\{v \in T_pM : \|v\| \leq \varepsilon\}$

Let $B(p,\varepsilon) = \{q \in M : d(p,q) \leq \varepsilon\}$.



Since $\partial B(p,\varepsilon)$ is compact:

$\exists\ y_1 \in \partial B(p,\varepsilon)$ which is the minimum of $d(\cdot, q)$ on $\partial B(p,a)$



Note that $y_1 = \exp_p(v)$ with $\|v\| = \varepsilon$

If $v_1 = \dfrac{v}{\|v\|}$.

Set

$$\gamma(t) = \exp_p(tv_1)$$ $\leftarrow$ It can be defined for all $t \in \mathbb{R}$

So that $\gamma(\varepsilon) = y_1$. Note: $\|\dot{\gamma}\| = 1$.

We want to show that $\gamma(t)$ passes through q at $t = d$.

Note by the triangle inequality: $d \leq \varepsilon + d(y_1, q) \Rightarrow d(y_1, q) \geq d - \varepsilon$.

But because it is the minimum: $d(y_1, q) \leq d - \varepsilon$ (compare with curve passing through)

~~We will show the stronger statement~~

Note that by the triangle inequality, $\forall\, s \in [0,d]$:

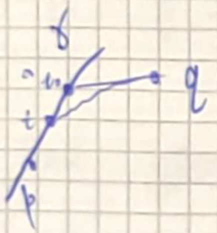
$$\underset{\underset{d}{\parallel}}{d(p,q)} \le d(p,\gamma(s)) + d(\gamma(s),q) \le s + d(\gamma(s),q)$$

$$\Rightarrow \quad d(\gamma(s),q) \ge d-s \quad \forall\, s \in [0,d] \quad ①$$

Let $J = \{ s \in [0,d] : d(\gamma(s),q) = d-s \}$. If we show that $d \in J$, we are done!

Note: 1) $0 \in J$, since $\gamma(0) = p$

2) J is an interval: If $t_0 \in J$, then $t \in J$ $\forall\, t \in [0,t_0]$,

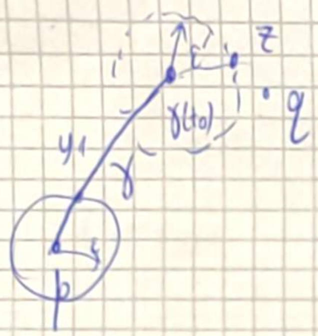


Since
$$d(\gamma(t),q) \le d(\gamma(t),\gamma(t_0)) + d(\gamma(t_0),q) \le t_0 - t + d - t_0 = d - t$$
so using ① ...

3) J is closed (by continuity).

~~4)~~ 4) $\varepsilon \in J$.

We will show that $J = [0,d]$: It suffices to show that if $t_0 < d \in J$, then J contains ~~an open neighborhood of t_0 of the form $(t_0-\varepsilon, t_0+\varepsilon)$~~ $[t_0, t_0+\varepsilon')$ for some $\varepsilon' > 0$



Let $t_0 \in J$, $\varepsilon < t_0 < d$. Like in the first step,

If $0 < \varepsilon' < \min\{ i(\gamma(t_0)), d(\gamma(t_0), q) \}$,

On the metric sphere $\partial B(\gamma(t_0), \varepsilon')$ I can find the point z which is closest to q.

As in the case of p and y_1:

If $z = \exp_{\gamma(t_0)}(w)$, $\|w\| = \varepsilon'$

and $\tilde{\gamma}(s) = \exp_{\gamma(t_0)}\left(\frac{w}{\|w\|} \cdot s\right)$, then $\gamma(\varepsilon') = z$

and $d(z, q) = d(\gamma(t_0), q) - \varepsilon' = d - t_0 - \varepsilon'$

Let $a(t) = \begin{cases} \gamma(t), & 0 \leq t \leq t_0 \\ \tilde{\gamma}(t - t_0), & t_0 \leq t \leq t_0 + \varepsilon' \end{cases}$

We will show that $a(t)$ ~~is locally length minimizing~~ coincide with $\gamma(t)$

Compute: $d(a(0), a(t_0+\varepsilon')) = d(p, z)$

$$\geq d(p,q) - d(z,q)$$
$$= d - (d - t_0 - \varepsilon')$$
$$= t_0 + \varepsilon'$$
$$= l(a)$$

So a achieves the minimum length $\Rightarrow$

$\Rightarrow$ a is a geodesic

$\overset{\text{uniqueness}}{\Rightarrow}$ $a = \gamma$

and $\gamma(t_0 + \varepsilon') = z \Rightarrow d(\gamma(t_0+\varepsilon'), q) = d(z,q) = d - (t_0 + \varepsilon')$

So $t_0 + \varepsilon' \in J$.

$\square$

Proof of the Hopf-Rinow theorem

a) (M, g) is geod. complete

b) $\exists p \in M$, $\Omega_p = T_pM$

c) Closed metric balls in (M,d) around p are compact.

d) (M,d) is complete

$a \Rightarrow b$: Obvious

$b \Rightarrow c$: From the previous proposition:

$$B(p,R) = \exp_p\{v \in T_pM : \|v\| \leq R\}$$

So it's the image through a continuous map of a compact set

$c \Rightarrow d$: Cauchy sequences are bounded

It remains to show $d \Rightarrow a$:

Let $\gamma : I \to \mathbb{R}$ be a maximal geodesic, $\|\dot\gamma\| = 1$,

$I = (t_-, t_+)$, $t_+ < +\infty$.

Assume that $0 \in I$. Then $\ell(\gamma|_{[0,t_+)}) = \int_0^{t_+} \|\dot\gamma\| \, dt = t_+ < +\infty$

So $\gamma|_{[0,t_+)}$ stays in a bounded set

~~completeness~~ $\Rightarrow$ completeness: If $t_n \in [0, t_+)$ is an increasing sequence with $t_n \to t_+$, then $\gamma(t_n)$ converges to $q \in M$.

↳ It has to be Cauchy

Since $d(\gamma(t_m), \gamma(t_n)) \leq |t_m - t_n|$

In local coordinates around p:

~~$\gamma(t_n)$ is a bounded sequence~~

$\gamma(t)$ is a solution to the 2^{nd} order ODE $\ddot\gamma^k + \Gamma^k_{\alpha\beta}(\gamma(t)) \dot\gamma^\alpha \dot\gamma^\beta = 0$ which stays bounded $\Rightarrow$ the maximal solution can be extended.

$\square$